Einstein's derivation of $E = mc^2$

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The goal is to understand Einstein's proof of the famous equation of rest energy of a body. I follow the original paper by Einstein [1].

0.1 Postulates of Special relativity

The two main postulates of Special relativity are losely stated as

- The physical laws are invariant under transition from one *inertial frame* to another, which physically means that conducting the same experiment in any inertial frame will yield the same results.
- There is an absolute constant $0 < c < \infty$ *[speed of light]*, such that any signal traveling with speed c with any given direction in a a given inertial frame, travels with the same speed in all other frames.

0.2 Deriving Lorentz transformation

Our goal in this section is to compute the transformation $L : (x, y, z, t) \mapsto (x', y', z', t')$ from orthogonal coordinates of a frame F to that of another frame F_v moving with constant velocity $v\mathbf{i}$ with respect to the F. First, from the first postulate of Special relativity together with First Newton's law, this transformation is linear. Indeed, by Newton's first law, a body moving freely, moves in a straight line, as this law is valid in F and F_v, then L should map straight lines to straight lines. Next, we can clearly set the line of motion of the spatial origin of F_v as the x-axis in F and set the zero time to be the same when the origins of F and F_v to coincide. We write the transformation as follows

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

Now because motion is along x-axis only, by an isotopy argument (which follows from Einstein's first postulate), one can assume that the plane $y = 0$ maps to $y' = 0$, and similarly $z = 0$ maps to $z' = 0$. So,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Because F_v moves with velocity v along $x-$axis with respect to F, a similar isotopy argument implies that the plane $x = vt$ should map to $x' = 0$. Hence $a_{11}v + a_{12}, a_{13}, a_{14} = 0$, therefore

$$x' = a_{11}(x - vt) \tag{1}$$

Now, we come to the next trick. Let F' and F'_v be the frames obtained by reversing x-axis, and z-axis of F and F_v. Observe that the transformation of F'_v to F' is the same as that from F to F_v, that's the transformations above should be invariant under:

$$x \leftrightarrow -x' \ , \ y \leftrightarrow y' \ , \ z \leftrightarrow -z' \ , \ t \leftrightarrow t'$$

Hence, using (1), we get

$$x = a_{11}(x' + vt') \tag{2}$$

It remains to determine a_{11}. We note that from the second postulate, $x = ct$ (the light signal in F) should map to $x' = ct'$ (The light signal in F_v). Substituting in (1), (2), we get

$$ct' = a_{11}(c - v)t$$

$$ct = a_{11}(c+v)t'$$

Now multiplying these two equations, we obtain $c^2tt' = a_{11}^2(c^2 - v^2)tt'$, hence

$$a_{11} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

we denote $a_{11} = \gamma$. Now solving (1) and (2) gives us

$$t' = \gamma\left(t + \frac{vx}{c^2}\right)$$

So the *lorentz-transformations* are

$$\boxed{x' = \gamma(x - vt)\ ,\ y' = y\ ,\ z' = z\ ,\ t' = \gamma\left(t + \frac{vx}{c^2}\right)}$$

0.3 Doppler effect.

Given a light wave in an inertial frame F, with frequency f. The frequency of the same wave as seen from inertial frame is changed, we let it be denoted by f'. There are many ways to calculate f', the simplest in my opinion is the following. Let $k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$ be the wave number of the wave (here λ is the wave length.) The wave may be described in frame F by $\sin(kx \pm 2\pi ft)$, and in frame F_v by $\sin(k'x' - 2\pi f't')$, then by comparing the arguments of sines and using Lorentz-transformations, we get

$$kx \pm 2\pi ft = k'x' \pm 2\pi f't' \implies kx \pm 2\pi ft = k'\gamma(x - vt) \pm 2\pi f'\gamma(t + \frac{vx}{c^2})$$

Hence comparing the coefficients of t, we get

$$\pm 2\pi f = -k'\gamma v \pm 2\pi f'\gamma \implies f = \gamma\left(1 \mp \frac{v}{c}\right)f' \tag{3}$$

Where the sign depends on the direction of the propagation of the wave.

0.4 The final step

Let a body of mass m be fixed at the origin point of frame F. Suppose it has a *rest Energy* E. Now suppose that the body emits two photons of equal energies $E_+ = hf_+$, $E_- = hf_-$ in the opposite direction ($+x-$axis and $-x-$axis respectively). After emmiting the two photons, suppose that the mass of the body decreased by Δm (which may be zero). Now due to conservation of energy the rest energy of the body becomes $E - (E_+ + E_-) = E - \Delta E$. Suppose that a frame F_v moves in the positive direction with velocity $v\mathbf{i}$ with respect to F. Now as seen from F_v, the photons have new energies (due to doppler effect (3)):

$$E'_+ = hf'_+ = h\gamma^{-1}\left(1 + \frac{v}{c}\right)f_+ = \gamma^{-1}\left(1 + \frac{v}{c}\right)\frac{\Delta E}{2}$$

and

$$E'_- = hf'_- = h\gamma^{-1}\left(1 - \frac{v}{c}\right)f_+ = \gamma^{-1}\left(1 - \frac{v}{c}\right)E_- = \gamma^{-1}\left(1 - \frac{v}{c}\right)\frac{\Delta E}{2}$$

(the energies now are not equal any more). We observe that;

- When $v << c$, we know that Newtonian laws are valid at such low velocities, that's the energy before the emission of photons is :

  $$E + \frac{1}{2}mv^2 + \epsilon(v/c), \tag{4}$$

  where $\epsilon(v/c) \to 0$ at $v/c \to 0$

- The energy after the emission is:

$$(E - \Delta E) + \frac{1}{2}(m - \Delta m)v^2 + E'_+ + E'_- \tag{5}$$

$$= (E - \Delta E) + \frac{1}{2}(m - \Delta m)v^2 + \gamma^{-1}\left(1 - \frac{v}{c}\right)\frac{\Delta E}{2} + \gamma^{-1}\left(1 + \frac{v}{c}\right)\frac{\Delta E}{2} \tag{6}$$

By Law of conservation of energies, we have (4) = (5), hence;

$$\frac{1}{2}\Delta m v^2 + \epsilon(\frac{v}{c}) = (1 - \gamma^{-1})\Delta E$$

Now taking $\frac{v}{c} \to 0$, we get:

$$\begin{aligned}
\Delta E &= \lim_{v/c \to 0} \frac{v^2}{2(1-\gamma^{-1})}\Delta m \\
&= \lim_{v/c \to 0} \frac{(v/c)^2}{2\left(1 - \sqrt{1 - v^2/c^2}\right)}(\Delta m)c^2 \\
&= \lim_{v/c \to 0} \frac{1 + \sqrt{1 - v^2/c^2}}{2}(\Delta m)c^2 \\
&= (\Delta m)c^2
\end{aligned}$$

References

[1] Karl von Meyenn. "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?" In: *Albert Einsteins Relativitätstheorie: Die grundlegenden Arbeiten* (1990), pp. 156–159.